1 Introduction

From the literature surrounding the development of beam theories, Saint-Venant (SV) approach continues to be considered as the reference to understand the beam structural behavior. This is due to the status or the asymptotic nature of (3D) SV’s solution as well as to the mechanical characteristics of the section provided by this SV’s solution (also called central solution).

Established first for homogeneous and isotropic beam with a loading limited to tractions acting on the end sections, SV’s solution has been extended by Iesan [1976] to any composite section wherein each material is anisotropic, taking also into account a lateral loading acting on the beam.

The sectional characteristics computed by CSction being those derived from SV’s solution, the purpose of this paper is:

• to recall the expression of SV’s solution;
• to specify the Saint-Venant beam-like theory that derives from SV’s solution;
• and to comment the validity of SV’s solution which is related to SV’s principle.

2 Saint-Venant’s problem extended to composite section

SV’s problem is a 3D equilibrium elastic problem (figure-1). The composite beam is along the z axis and is occupying a prismatic domain V of a constant cross section S and length L. \( S_{lat} \) is the lateral surface and \( S_0 \) and \( S_L \) are the end cross sections. \( M = G + GM \) denotes a point of V where G is the section center. The beam is submitted only to the tractions \( H_0 \) and \( H_L \) acting on \( S_0 \) and \( S_L \), respectively. The materials of the cross section are anisotropic and perfectly bonded together.

The equations of the linearized equilibrium are:
\[
\begin{align*}
\text{div } \sigma &= 0 \quad \text{in } V \\
\varepsilon(\xi) &= \frac{1}{2}(\nabla^t \xi + \nabla \xi) \quad \text{in } V \\
\sigma &= \mathbf{K} : \varepsilon(\xi) \quad \text{in } V \\
\sigma(n) &= 0 \quad \text{on } S_{\text{lat}} \\
\sigma(-z) &= H_0 \quad \text{on } S_0 \\
\sigma(z) &= H_L \quad \text{on } S_L
\end{align*}
\] (1)

where \(\varepsilon(\xi)\) is the strain tensor associated to the displacement field \(\xi\); \(\nabla, (\cdot)^t\) and \((\cdot)\) denote the gradient, the transpose and the double contraction operators, respectively; \(\mathbf{K}\) denotes the elasticity tensor, \(\sigma\) is the stress tensor and \(n\) is the unit vector that is normal and external to \(S_{\text{lat}}\).

3 Saint-Venant’s solution

Let \([R, M]\) denote the 1D cross-sectional stresses defined by

\[
R = \int_S \sigma(z) \, dS \quad M = \int_S GM \wedge \sigma(z) \, dS
\]

Their components with respect to \((x, y, z)\) and denoted by \(X_i \in \{T_x, T_y, N, M_x, M_y, M_t\}\) are the 6 classical internal forces: respectively, the \(2\) shear forces, the axial force, the \(2\) bending moments and the tossional moment.

3D SV’s solution is the unique \((z\text{-polynomial})\) solution that exactly satisfies eqs-1 and satisfies the boundary conditions eqs-2 only in terms of resultant (force and moment).

\[
\begin{align*}
\int_{S_0} \sigma(-z) \, dS &= \int_{S_0} H_0 \, dS \quad \int_{S_0} GM \wedge \sigma(-z) \, dS = \int_{S_0} GM \wedge H_0 \, dS \\
\int_{S_L} \sigma(z) \, dS &= \int_{S_L} H_L \, dS \quad \int_{S_L} GM \wedge \sigma(z) \, dS = \int_{S_L} GM \wedge H_L \, dS
\end{align*}
\] (3)

The expression of 3D SV’s solution is given by:

\[
\begin{align*}
\xi_{sv}(x, y, z) &= u(z) + \omega(z) \wedge GM + \sum_{i=1}^{6} X_i(z) U_i(x, y) \\
\sigma_{sv}(x, y, z) &= \sum_{i=1}^{6} X_i(z) \sigma_i(x, y)
\end{align*}
\] (4)

\[
\begin{align*}
\left[ \begin{array}{c} R' \\ M' + z \wedge R \\
\end{array} \right] &= \left[ \begin{array}{c} 0 \\
\end{array} \right] \quad \text{sur}[0, L] \\
\left[ \begin{array}{c} \gamma \\ \chi \\
\end{array} \right] &= \left[ \begin{array}{c} u' + z \wedge \omega \\ \omega' \\
\end{array} \right] = \Lambda \left[ \begin{array}{c} R \\ M \\
\end{array} \right] \\
[R, M]_0 &= [-F_0, C_0] \\
[R, M]_L &= [F_L, C_L]
\end{align*}
\] (6)
In eqs.4-5, the (1D) quantities \( \{u, \omega, R, M\} \) are solution of the (1D) equations eqs.6-7-8 where
\[
F_{0,L} = \int_{S_0,L} H_{0,L} \, dS \quad \text{et} \quad C_{0,L} = \int_{S_0,L} GM \wedge H_{0,L} \, dS
\]
\((\cdot)’\) denotes the derivative with respect to \( z \).
Besides, the (6×6) compliance operator \( \Lambda \), is related to the elasticity tensor \( K \) by:
\[
\Lambda = [\lambda_{ij}] \quad ; \quad \lambda_{ij} = \int_S \sigma^i : K^{-1} \sigma^j \, dS \quad \text{for } (i, j) \in \{1, \ldots, 6\}
\]

### 3.1 Mechanical characteristics of the section

An important property of SV’s solution is that the quantities \( \{\Lambda, U^i, \sigma^i\} \) depend only on the section nature (shape and materials):

- \( \Lambda \) is the full (6×6) compliance matrix of the 1D structural behavior of the beam; in this matrix, the off-diagonal terms express all the elastic couplings between extensional, flexural and torsional deformations that can occur for an arbitrary composite section.

- \( U^i \) are displacements that describe the sectional deformability related to each internal force \( X_i \); one can split this displacements in two parts:
  
  - \( [U^i_x x + U^i_y y] \) describe the Poisson’s effects
  - \( [U^i_z z] \) describe the warping

- \( \sigma^i \) give the 3D sectional stress field related to each internal force \( X_i \).

**Important**

- These sectional characteristics are fundamental to understand the mechanical behavior of the section and hence the structural behavior of the beam. They can be computed once for all and used later, for a given beam problem. In CSsection, this is performed using the numerical method proposed by El Fatmi and Zenzi [2002]; this method consists in solving by 2D finite elements a set of elastic problems defined on the section.

- These sectional characteristics \( \{\Lambda, U^i, \sigma^i\} \) constitute a relevant set of mechanical information on the behavior of the composite section that can really help R&D engineers to design and optimize composite sections and composite beams.

### 3.2 Saint-Venant beam-like theory

The following set of 1D equations that derive from the expression of SV’s solution
\[
\begin{align*}
R' &= 0 \\
M' + z \wedge R &= 0
\end{align*}
\]
\[
[\gamma] = \Lambda \begin{bmatrix} R \\ M \end{bmatrix} \quad [R, M]_{0,L} = [F, C]_{0,L}
\]
define what we mean by Saint-Venant (1D) beam-like theory.
Let us come back to 3D SV’s problem. As soon as the sectional characteristics \( \{\Lambda, U^i, \sigma^i\} \) are known, one can get the 3D SV’s solution in two steps:
1. Λ is used to solve the 1D problem defined by eqs.10;
2. the 1D solution \{u(z), \omega(z), R(z), M(z)\} of (10) allows then to generate 3D SV’s solution conforming to eqs-4-5. Note that this step is straightforward and does not need any further computation.

4. Saint-Venant’s principle and the validity of Saint-Venant’s solution

Let us consider for these purposes the cantilever beam subjected to a traction applied on the free end (figure-2).

3D SV-Solution

edge effect interior solution edge effect

Figure 2: Saint-Venant’s solution and end effects.

Moving away from the ends, the end-effects vanish after a certain distance denoted by \(d\) in figure-3. SV’s principle is rather vague; it only indicates a trend. However, SV’s principle is usually taken to mean that end-effects vanish closely to the ends \((d \ll L)\); in many cases, this is false. End-effects depend on the nature of the section (shape and materials) and the boundary conditions.

3D SV-Solution

\(d\) \(d\)

Figure 3: Interior solution and end-effects

When strongly anisotropic materials and/or thin walled open section are of concern, end-effects can persist over distances comparable to the beam length \((d \approx L)\). In that case, SV’s solution no longer represents the solution in the interior part of the beam.

The most famous case deals with the torsion of a cantilever thin-walled open profile: the built-in effect (due to the restrained warping) can reach the loaded end, leading to a beam mechanical behavior significantly different from that predicted by Saint-Venant’s torsion (the solution is to use Vlasov’s torsion [Vlasov, 1961] instead of Saint-Venant’s torsion).

This example is edifying to understand the influence of the section nature (shape and materials) on the extent of the edge-effect. In figure-4, the built-in effect is considered for three kinds of section:
for the homogeneous square section, edge effect vanishes closely to the built-in section ($d \ll L$); SV’s solution can describe the interior solution and hence the corresponding beam-like theory can correctly describe the 1D structural behavior of the beam;

for the homogeneous open walled section, the edge effect extends to the interior part of the beam and can reach the loaded end; in this case SV’s solution can no longer describe the interior solution;

for the composite section made with two isotropic materials: if ($E_1 = E_2$) the section behavior corresponds to the square section; if ($E_1 \gg E_2$) the section behavior corresponds to the open walled section; it is then clear that $d$ will depend on the ratio ($\frac{E_1}{E_2}$) for this composite section; in this case what about SV’s solution?

What to learn from this example?

- The behavior of a composite beam is the result of the whole nature of the section: shape-and-materials.

- For particular section (shape and materials) and common beam slenderness, it is important to improve the prediction of beam theories: the classical ones (Bernoulli, Timoshenko) and also the Saint-Venant beam-like theory. This refinement has to incorporate at least, the most influential end-effects.

5 Saint-Venant’s solution extended to lateral loading

5.1 Extended Saint-Venant’s problem and solution

This extension (figure-5) consists in considering two supplementary $z$-constant loads: body force density $f^d$ acting on the beam and surface force density $F^d$ acting on the lateral surface $S_{lat}$. In that case SV’s solution is given by:
Figure 5: Saint-Venant’s problem extended to lateral loads

\[
\xi^{sv}(x, y, z) = u(z) + \omega(z) \land GM + \sum_{i=1}^{6} X_i(z) U^i(x, y) + W^d(x, y) \tag{11}
\]

\[
\sigma^{sv}(x, y, z) = \sum_{i=1}^{6} X_i(z) \sigma^i(x, y) + \sigma^d(x, y) \tag{12}
\]

\[
\begin{cases}
R' + p^d = 0 \\
M' + z \land R + \mu^d = 0
\end{cases}
| \text{sur[0,L]} \tag{13}
\]

\[
\begin{bmatrix}
\gamma \\
\chi
\end{bmatrix} = \begin{bmatrix}
\omega' + z \land \omega' \\
\chi'
\end{bmatrix} = \Lambda \begin{bmatrix}
R \\
M
\end{bmatrix} + \begin{bmatrix}
\gamma^d \\
\chi^d
\end{bmatrix} \tag{14}
\]

with

\[
p^d = \int_S f^d dS + \int_\Gamma F^d d\Gamma \quad \mu^d = \int_S GM \land f^d dS + \int_\Gamma GM \land F^d d\Gamma
\]

where \( \Gamma \) is a part of the contour where \( F^d \) is applied.
The vector \( W^d \) and the tensor \( \sigma^d \) depend on the section nature (shape and materials) and the loading \([f^d, F^d]\); the dependency of \( W^d \) and \( \sigma^d \) on the loading \([f^d, F^d]\) is linear.

5.2 Distortion modes for the section

The way used by CSSection to characterize the distortions of a thin/thick walled section consists in considering several cases of loads \( F^d_k (k=1,...,n) \) (figure-6) and to compute the corresponding \( W^d_k \).

The choice of a load case is designed to cause local bending in a branch of the section contour; each case \( W^d_k \) leads to a sectional deformation in the plane of the section which is stored as distortion mode for the section. For example (figure-6), five loads are considered to compute five distortion modes for the section.

References

Figure 6: Five cases of uniformly distributed force to characterize the sectional distortions.
